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Thermal stresses around a crack in the nonhomogeneous interfacial layer between two dissimilar elastic half-planes

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Abstract

Thermal stresses around a crack in the interfacial layer between two dissimilar elastic half-planes are solved. The surfaces of the crack are assumed to be insulated. The material constants of the interfacial layer are assumed to vary continuously from those of the upper half-plane to those of the lower half-plane. Uniform heat flows perpendicular the crack. Stress intensity factors are calculated numerically for several thicknesses of the interfacial layer.

Keywords: Thermal stress; Crack; Heat flow; Nonhomogeneous interfacial layer; Ceramics

1. Introduction

Ceramic materials have good anti-oxidation characteristics at high temperatures and are good thermal insulators. Coating metal components with ceramics is one way of increasing the application range of metal parts. If a diffusion method is used to join a ceramic and a metal, a thin diffusion layer appears between ceramic and the metal. The diffusion layer is relatively weak and inclined to fracture. It is therefore necessary to solve the stress intensity factors around a crack in the interfacial zone in order to avoid catastrophic failure in composite materials.

Delale and Erdogan (1988) considered that the cracked layer is nonhomogeneous and its material constants vary continuously within a range from those of the upper half-plane to those of the lower half-plane. The two-dimensional stress and displacement fields are solved for internal pressure on the crack surfaces. Later, axisymmetric solutions have been determined for a penny-shaped crack in an interfacial nonhomogeneous layer between two dissimilar elastic half-spaces by Ozturk and Erdogan (1995, 1996). Xue-Li and Duo (1996) provided the stresses around a cylindrical crack in a nonhomogeneous layer between an infinite elastic medium and a circular elastic cylinder under Mode III torsional loading.

As for Mode I loading, axisymmetric stresses were solved for a cylindrical crack in an interfacial cylindrical layer between a circular elastic cylinder and an infinite elastic medium (Itou and Shima, 1999).

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The material constants vary continuously from those of the cylinder to those of the infinite medium. Rather than apply the nonhomogeneous theory of elasticity to solve the problem, the nonhomogeneous layer is divided into several homogeneous layers each having different elastic constants. The method has also been applied to obtain dynamic stresses around a crack in a nonhomogeneous interfacial layer between two dissimilar elastic half-planes (Itou, 2001a). Dynamic stresses for a penny-shaped crack in a functionally graded material (FGM) interlayer between two dissimilar homogeneous half-spaces are solved under Mode III torsional loading (Li and Weng, 2002).

In the present paper, thermal stresses are solved for a crack in the interfacial layer between two dissimilar elastic half-planes. Heat flows perpendicular to the crack and is interrupted by the heat insulating surfaces of the crack. The material constants in the layer vary continuously from those of the upper half-plane to those of the lower half-plane. In order to solve the problem, the same method employed in Itou and Shima (1999) and Itou (2001a) is applied. Namely, the nonhomogeneous layer is divided into several homogeneous layers that have different elastic constants. The solution is obtained using expressions in the theory of orthotropic elasticity. Numerical calculations are carried out only for composite materials made of isotropic materials. However, whenever solutions are needed for composite materials made of orthotropic materials, these can be calculated numerically.

Thermal stresses were solved for an infinite orthotropic plate weakened by a crack by Tsai (1984). The corresponding problem for an infinite orthotropic cracked layer was studied by Itou (2000). Recently, the author provided thermal stresses around two parallel cracks in an infinite orthotropic plate (Itou, 2001b).

In the present solution, the temperature field is determined by reducing the boundary conditions to a set of dual integral equations using the Fourier transform technique. If a difference in the crack surface temperature is expanded in a series of functions that are zero outside the crack, the continuous temperature condition outside the crack is satisfied. Unknown coefficients in the series are then determined using the Schmidt method (Morse and Feshbach, 1958) so as to satisfy the condition inside the crack. Next, the mixed boundary value conditions related to displacements and stresses are reduced to two sets of dual integral equations. To solve the equations, differences in displacements at the crack are also expanded in a series of functions that are zero outside the crack. The Schmidt method (Yau, 1967) is once more used to obtain the unknown coefficients in the series.

As the number of divisions in the interfacial layer increases, the solution approaches that for a nonhomogeneous interfacial layer. Using this approach, the stress intensity factors are computed numerically for composite materials made of ceramic and steel.

2. Fundamental equation

Consider a crack located on the x-axis from $-a$ to $+a$ with respect to the rectangular co-ordinates (x, y) as shown in Fig. 1. The interfacial layer (A) is denoted by $(-H_B \leqslant y \leqslant H_C)$. The upper half-plane and the lower half-plane are labeled half-plane (C) and half-plane (B), respectively. The interfacial layer (A) is further divided into an upper interfacial layer (A-1) denoted by $(0 \leqslant y \leqslant H_C)$ and a lower interfacial layer (A-2) denoted by $(-H_B \leqslant y \leqslant 0)$. If a state of plane stress is assumed, the stresses can be expressed by

$$\tau_{xx} = Q_{11}\varepsilon_{xx} + Q_{12}\varepsilon_{yy} - \beta_1 T, \quad \tau_{yy} = Q_{12}\varepsilon_{xx} + Q_{22}\varepsilon_{yy} - \beta_2 T, \quad \tau_{xy} = Q_{66}\gamma_{xy} \tag{1}$$

with

$$\begin{aligned} &Q_{11} = E_{xx}/(1 - v_{yx}v_{xy}), \quad Q_{22} = E_{yy}/(1 - v_{xy}v_{yx}) \\ &Q_{12} = E_{yy}v_{xy}/(1 - v_{yx}v_{xy}) = E_{xx}v_{yx}/(1 - v_{xy}v_{yx}) \\ &Q_{66} = G_{xy}, \quad \beta_1 = Q_{12}\alpha_{yy} + Q_{11}\alpha_{xx}, \quad \beta_2 = Q_{12}\alpha_{xx} + Q_{22}\alpha_{yy} \end{aligned} \tag{2}$$

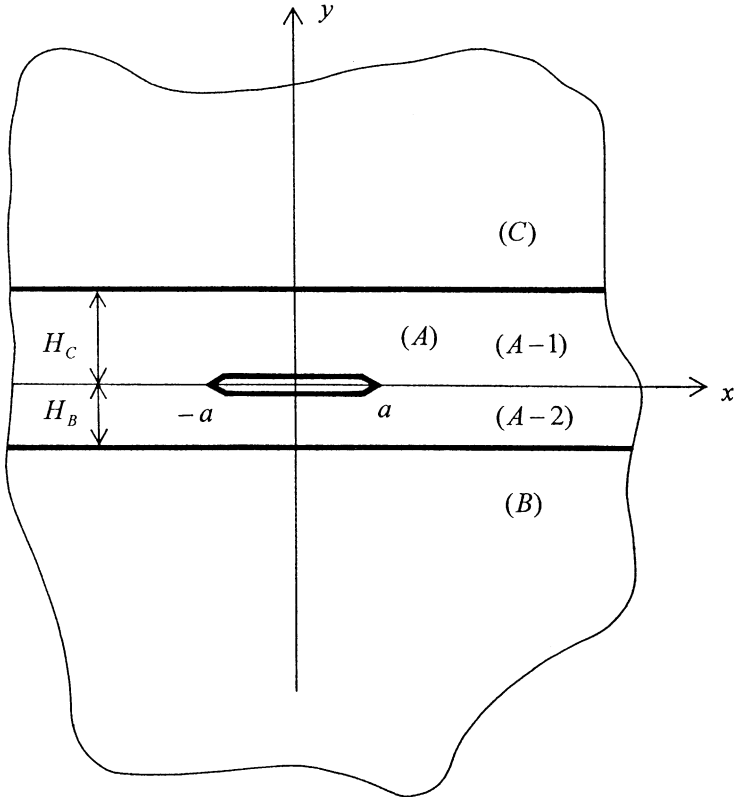

Fig. 1. Geometry and co-ordinate system.

where the strain–displacement relations are as follows:

$$\varepsilon_{xx} = \partial u/\partial x, \quad \varepsilon_{yy} = \partial v/\partial y, \quad \gamma_{xy} = \partial u/\partial y + \partial v/\partial x \tag{3}$$

and E_{xx}, E_{yy} are the Young's moduli, G_{xy} is the shear modulus, v_{xy}, v_{yx} are the Poisson ratios, and α_{xx}, α_{yy} are the coefficients of linear thermal expansion. In Eq. (1), the temperature T satisfies the following partial differential equation

$$\partial^2 T/\partial x^2 + k^2 \partial^2 T/\partial y^2 = 0 \tag{4}$$

with

$$k^2 = k_y/k_x \tag{5}$$

where k_y, k_x are the thermal conductivities.

The equations of equilibrium can be written as

$$\begin{aligned} &Q_{11}\partial^2 u/\partial x^2 + Q_{66}\partial^2 u/\partial y^2 + L\partial^2 v/\partial x\partial y - \beta_1 \partial T/\partial x = 0 \\ &Q_{66}\partial^2 v/\partial x^2 + Q_{66}\partial^2 v/\partial y^2 + L\partial^2 u/\partial x\partial y - \beta_2 \partial T/\partial y = 0 \end{aligned} \tag{6}$$

with

$$L = Q_{12} + Q_{66}. \tag{7}$$

The material property E_{xx} probably varies continuously in the interfacial layer with respect to y, as shown in Fig. 2. Other constants $(G_{xy}, v_{xy}, v_{yx}, \alpha_{xx}, \alpha_{yy}, k_y, k_x)$ are assumed to vary in a similar manner according to the curve describing E_{xx}. The Young's modulus E_{yy} is expressed by the relation

$$E_{yy} = (v_{yx}/v_{xy})E_{xx}. \tag{8}$$

Consider the case in which the heat flux q flows in the direction of the negative y-axis. By restricting our concern to only the stress intensity factors, it is possible to solve the problem for the following boundary conditions

$$k_{yA1}\partial T_{A1}/\partial y = k_{yA2}\partial T_{A2}/\partial y \quad \text{for } y = 0,\ |x| < \infty \tag{9}$$

$$\partial T_{A1}/\partial y = -q/k_{yA1} = -t \quad \text{for } y = 0,\ |x| < a \tag{10}$$

$$T_{A1} = T_{A2} \quad \text{for } y = 0,\ a < |x| \tag{11}$$

$$k_{yC}\partial T_C/\partial y = k_{yA1}\partial T_{A1}/\partial y, \quad T_C = T_{A1} \quad \text{for } y = H_C,\ |x| < \infty \tag{12}$$

$$k_{yA2}\partial T_{A2}/\partial y = k_{yB}\partial T_B/\partial y, \quad T_{A2} = T_B \quad \text{for } y = -H_B,\ |x| < \infty \tag{13}$$

$$\tau_{yyA1} = \tau_{yyA2}, \quad \tau_{xyA1} = \tau_{xyA2} \quad \text{for } y = 0,\ |x| < \infty \tag{14}$$

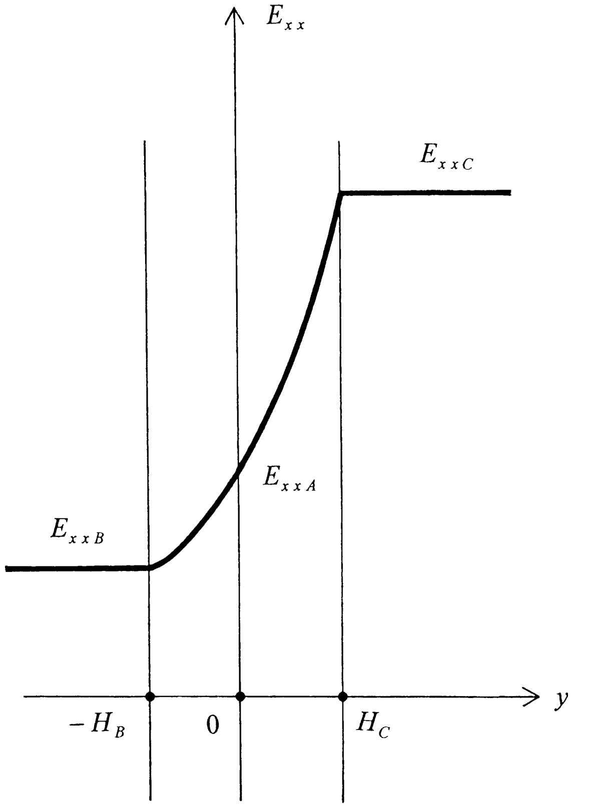

Fig. 2. Young's modulus E_{xx} as a function of y.

$$\tau_{yyA1} = 0, \quad \tau_{xyA1} = 0 \quad \text{for } y = 0, \ |x| < a \tag{15}$$

$$u_{A1} = u_{A2}, \quad v_{A1} = v_{A2} \quad \text{for } y = 0, \ a < |x| \tag{16}$$

$$\tau_{yyC} = \tau_{yyA1}, \quad \tau_{xyC} = \tau_{xyA1}, \quad u_C = u_{A1}, \quad v_C = v_{A1} \quad \text{for } y = H_C, \ |x| < \infty \tag{17}$$

$$\tau_{yyA2} = \tau_{yyB}, \quad \tau_{xyA2} = \tau_{xyB}, \quad u_{A2} = u_B, \quad v_{A2} = v_B \quad \text{for } y = -H_B, \ |x| < \infty \tag{18}$$

where t is a constant and the variables with subscripts A1, A2, B, C are those for the layers (A-1), (A-2), the upper half-plane (C) and the lower half-plane (B), respectively. It is assumed that the crack faces do not come into contact and also that the crack surfaces are thermally insulated.

3. Division of the interfacial layer

In order to solve the problem of thermal stresses in the nonhomogeneous layer, the interfacial layer (A) in Fig. 1 is first replaced by several homogeneous layers. The number of homogeneous layers, m, must be odd. In the present example, m is set to 3. It should be noted that if $m = 3$, the interfacial layer (A) is divided into four layers because the cracked layer denoted by $(-H_2 \leqslant y \leqslant H_1)$ separates into two parts. Namely, the interfacial layer (A) is divided into layer (1) occupying $(0 \leqslant y \leqslant H_1)$, layer (3) occupying $(H_1 \leqslant y \leqslant H_3)$, layer (2) occupying $(-H_2 \leqslant y \leqslant 0)$ and layer (4) occupying $(-H_4 \leqslant y \leqslant -H_2)$, as shown in Fig. 3. For convenience, the upper half-plane (C) and the lower half-plane (B) are denoted by (5) and (6), respectively.

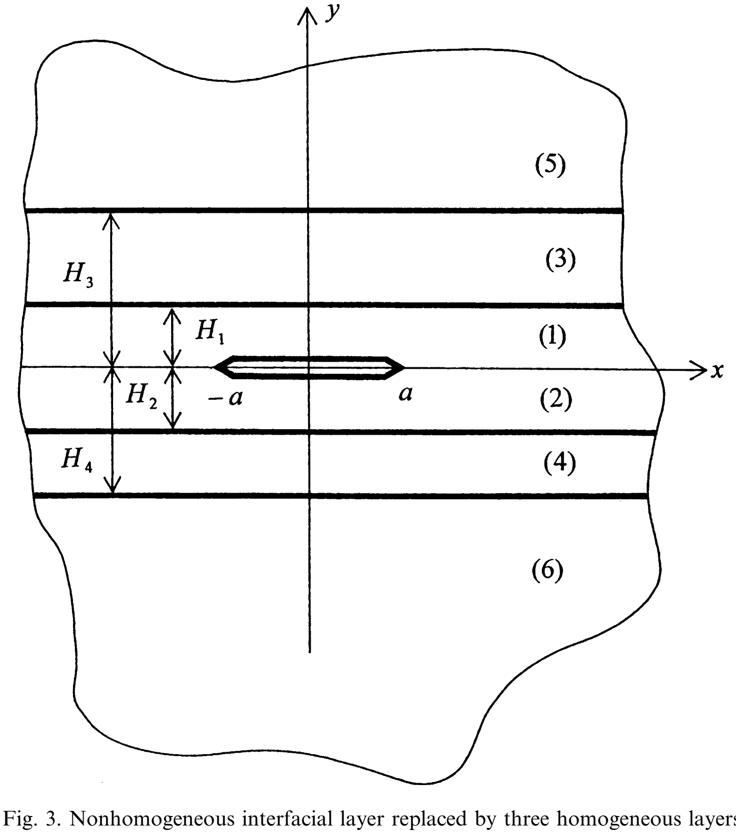

Fig. 3. Nonhomogeneous interfacial layer replaced by three homogeneous layers.

In the interfacial layer (A), all material properties except for E_{yy} are assumed to vary linearly with respect to y. For example, E_{xxA} is expressed as

$$E_{xxA} = E_{xxB} + (E_{xxC} - E_{xxB})(y + H_B)/(H_C + H_B). \tag{19}$$

The material properties $(G_{xy}, \nu_{xy}, \nu_{yx}, \alpha_{xx}, \alpha_{yy}, k_y, k_x)$ are also given by similar forms like in Eq. (19). Of course, E_{yy} is expressed by Eq. (8). For $m = 3$ the Young's moduli of the layers (1)–(4) take the average values $E_{xx1}, E_{xx2}, E_{xx3}, E_{xx4}$ as seen in Fig. 4 instead of E_{xxA} denoted by Eq. (19). The same applies to the other material properties.

For $m = 3$, the boundary conditions (9)–(18) can be replaced by the following equations:

$$k_{y1}\partial T_1/\partial y = k_{y2}\partial T_2/\partial y \quad \text{for } y = 0,\ |x| < \infty \tag{20}$$

$$\partial T_1/\partial y = -t \quad \text{for } y = 0,\ |x| < a \tag{21}$$

$$T_1 = T_2 \quad \text{for } y = 0,\ a < |x| \tag{22}$$

$$k_{y3}\partial T_3/\partial y = k_{y1}\partial T_1/\partial y, \quad T_3 = T_1 \quad \text{for } y = H_1,\ |x| < \infty \tag{23}$$

$$k_{y5}\partial T_5/\partial y = k_{y3}\partial T_3/\partial y, \quad T_5 = T_3 \quad \text{for } y = H_3,\ |x| < \infty \tag{24}$$

$$k_{y2}\partial T_2/\partial y = k_{y4}\partial T_4/\partial y, \quad T_2 = T_4 \quad \text{for } y = -H_2,\ |x| < \infty \tag{25}$$

$$k_{y4}\partial T_4/\partial y = k_{y6}\partial T_6/\partial y, \quad T_4 = T_6 \quad \text{for } y = -H_4,\ |x| < \infty \tag{26}$$

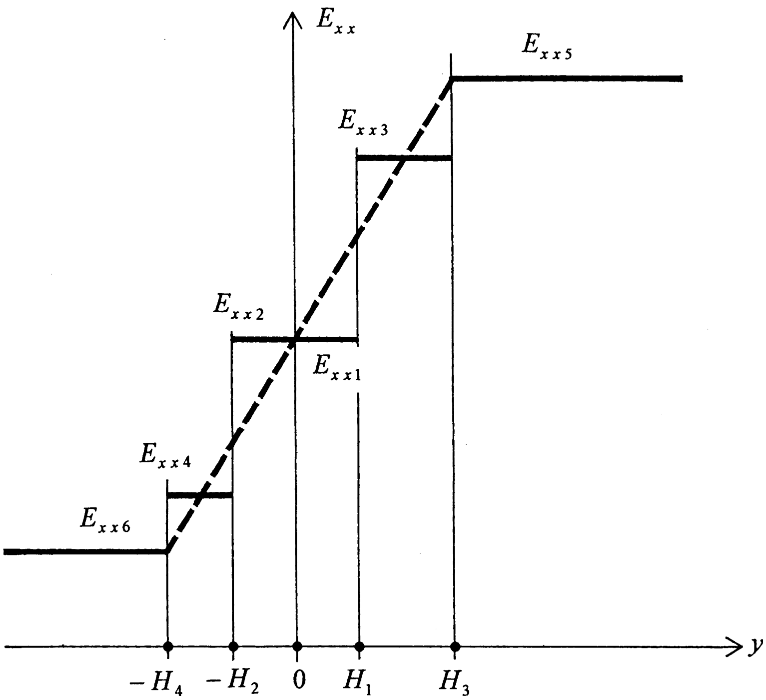

Fig. 4. Young's modulus E_{xx} in homogeneous sublayers.

$$\tau_{yy1} = \tau_{yy2}, \quad \tau_{xy1} = \tau_{xy2} \quad \text{for } y = 0, \ |x| < \infty \tag{27}$$

$$\tau_{yy1} = 0, \quad \tau_{xy1} = 0 \quad \text{for } y = 0, \ |x| < a \tag{28}$$

$$u_1 = u_2, \quad v_1 = v_2 \quad \text{for } y = 0, \ a < |x| \tag{29}$$

$$\tau_{yy3} = \tau_{yy1}, \quad \tau_{xy3} = \tau_{xy1}, \quad u_3 = u_1, \quad v_3 = v_1 \quad \text{for } y = H_1, \ |x| < \infty \tag{30}$$

$$\tau_{yy5} = \tau_{yy3}, \quad \tau_{xy5} = \tau_{xy3}, \quad u_5 = u_3, \quad v_5 = v_3 \quad \text{for } y = H_3, \ |x| < \infty \tag{31}$$

$$\tau_{yy2} = \tau_{yy4}, \quad \tau_{xy2} = \tau_{xy4}, \quad u_2 = u_4, \quad v_2 = v_4 \quad \text{for } y = -H_2, \ |x| < \infty \tag{32}$$

$$\tau_{yy4} = \tau_{yy6}, \quad \tau_{xy4} = \tau_{xy6}, \quad u_4 = u_6, \quad v_4 = v_6 \quad \text{for } y = -H_4, \ |x| < \infty \tag{33}$$

4. Analysis

To find the solutions, we introduce the Fourier transforms

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x)\exp(\mathrm{i}\xi x)\,\mathrm{d}x \tag{34}$$

$$f(x) = 1/(2\pi)\int_{-\infty}^{\infty} \bar{f}(\xi)\exp(-\mathrm{i}\xi x)\,\mathrm{d}\xi \tag{35}$$

Applying Eq. (34) into Eq. (6) results in

$$\begin{aligned} &Q_{66}\mathrm{d}^2\bar{u}/\mathrm{d}y^2 - \xi^2 Q_{11}\bar{u} - \mathrm{i}L\xi\mathrm{d}\bar{v}/\mathrm{d}y + \mathrm{i}\beta_1\xi\bar{T} = 0 \\ &Q_{22}\mathrm{d}^2\bar{v}/\mathrm{d}y^2 - \xi^2 Q_{66}\bar{v} - \mathrm{i}L\xi\mathrm{d}\bar{u}/\mathrm{d}y - \beta_2\partial\bar{T}/\partial y = 0 \end{aligned} \tag{36}$$

Eliminating $\bar{u}$ or $\bar{v}$ from Eq. (36), the ordinary differential equations are obtained

$$\begin{aligned} &\zeta_1\mathrm{d}^4\bar{u}/\mathrm{d}y^4 + \zeta_2\mathrm{d}^2\bar{u}/\mathrm{d}y^2 + \zeta_3\bar{u} = \mathrm{i}\eta_1\mathrm{d}^2\bar{T}/\mathrm{d}y^2 + \mathrm{i}\eta_2\bar{T} \\ &\zeta_1\mathrm{d}^4\bar{v}/\mathrm{d}y^4 + \zeta_2\mathrm{d}^2\bar{v}/\mathrm{d}y^2 + \zeta_3\bar{v} = \eta_3\mathrm{d}^3\bar{T}/\mathrm{d}y^3 + \eta_4\mathrm{d}\bar{T}/\mathrm{d}y \end{aligned} \tag{37}$$

with

$$\begin{aligned} &\zeta_1 = Q_{22}Q_{66}/L, \quad \zeta_2 = -\xi^2(Q_{66}^2 + Q_{11}Q_{22} - L^2)/L, \quad \zeta_3 = Q_{11}Q_{66}\xi^4/L \\ &\eta_1 = \xi(\beta_2 - \beta_1 Q_{22}/L), \quad \eta_2 = \xi^3\beta_1 Q_{66}/L, \quad \eta_3 = Q_{66}\beta_2/L \\ &\eta_4 = \xi^2(\beta_1 - \beta_2 Q_{11}/L) \end{aligned} \tag{38}$$

The Fourier-transformed stresses are found to be

$$\begin{aligned} &\bar{\tau}_{xx} = Q_{11}(-\mathrm{i}\xi)\bar{u} + Q_{12}\mathrm{d}\bar{v}/\mathrm{d}y - \beta_1\bar{T}, \quad \bar{\tau}_{yy} = Q_{12}(-\mathrm{i}\xi)\bar{u} + Q_{22}\mathrm{d}\bar{v}/\mathrm{d}y - \beta_2\bar{T} \\ &\bar{\tau}_{xy} = Q_{66}\mathrm{d}\bar{u}/\mathrm{d}y - \mathrm{i}\xi Q_{66}\bar{v} \end{aligned} \tag{39}$$

Eq. (4) can now be expressed in the Fourier domain as

$$\mathrm{d}^2\bar{T}/\mathrm{d}y^2 - (\xi/k)^2\bar{T} = 0. \tag{40}$$

First, the temperature field T is solved. The solutions of the Eq. (40) have the following form for each layer i ($i = 1, 2, 3, 4$):

$$\bar{T}_i = A_i \sinh(|\xi|y/k_i) + B_i \cosh(|\xi|y/k_i) \tag{41}$$

where A_i and B_i are unknown coefficients. For the upper half-plane (5) and lower the half-plane (6), $\bar{T}_5$ and $\bar{T}_6$ are expressed by

$$\bar{T}_5 = A_5 \exp(-|\xi|y/k_5) \tag{42}$$

$$\bar{T}_6 = A_6 \exp(-|\xi|y/k_6) \tag{43}$$

where A_5 and A_6 are also unknown coefficients.

Using Eqs. (20),(23)–(26), the unknowns B_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, A_6 can be expressed by an unknown A_1 as follows,

$$\begin{aligned}
&B_1 = f_1(\xi)A_1, \quad A_2 = f_2(\xi)A_1, \quad B_2 = f_3(\xi)A_1, \quad A_3 = f_4(\xi)A_1 \\
&B_3 = f_5(\xi)A_1, \quad A_4 = f_6(\xi)A_1, \quad B_4 = f_7(\xi)A_1, \quad A_5 = f_8(\xi)A_1 \\
&A_6 = f_9(\xi)A_1
\end{aligned} \tag{44}$$

where the expressions of the known functions $f_i(\xi)$ $(i = 1, 2, \ldots, 9)$ are shown in Appendix A.

To satisfy Eq. (22), the temperature difference at $y = 0$ is expanded by the series

$$\begin{aligned}
&\pi(T_1^0 - T_2^0) = \sum_{n=1}^{\infty} c_n \cos[(2n-1)\sin^{-1}(x/a)] \quad \text{for } |x| < a \\
&\pi(T_1^0 - T_2^0) = 0 \quad \text{for } a < |x|
\end{aligned} \tag{45}$$

where c_n are unknown coefficients and the superscript 0 denotes the values at $y = 0$. The Fourier transform of Eq. (45) now becomes

$$\pi\left(\bar{T}_1^0 - \bar{T}_2^0\right) = \sum_{n=1}^{\infty} c_n[(2n-1)/\xi]J_{2n-1}(a\xi) \tag{46}$$

where $J_n(\xi)$ is the Bessel function. The left-hand side in Eq. (46) can be represented by

$$\pi\left(\bar{T}_1^0 - \bar{T}_2^0\right) = b_1(\xi)A_1 \tag{47}$$

with

$$b_1(\xi) = f_1(\xi) - f_3(\xi). \tag{48}$$

From Eqs. (46) and (47), it can be seen that the unknown A_1 can be replaced by c_n in the following manner

$$A_1 = \sum_{n=1}^{\infty} c_n\{(2n-1)/[b_1(\xi)\xi]\}J_{2n-1}(a\xi). \tag{49}$$

Then, the Fourier transform of the temperature gradient $\partial\bar{T}_1/\partial y$ at $y = 0$ can be represented by

$$\partial\bar{T}_1/\partial y = \sum_{n=1}^{\infty} c_n\{(2n-1)|\xi|/[k_1 b_1(\xi)\xi]\}J_{2n-1}(a\xi). \tag{50}$$

Eq. (21) is the remaining boundary condition with respect to the temperature field and it can now be reduced to the form

$$\sum_{n=1}^{\infty} c_n F_n(x) = -t, \quad \text{for } 0 \leqslant x < a \tag{51}$$

with

$$F_n(x) = [(2n-1)/(k_1\pi)]\left\{\int_0^{\infty}[b_2(\xi) - b_2^L]J_{2n-1}(a\xi)\cos(\xi x)\,\mathrm{d}\xi + b_2^L\cos[(2n-1)\sin^{-1}(x/a)]/(a^2-x^2)^{1/2}\right\} \tag{52}$$

and

$$b_2(\xi) = 1/b_1(\xi). \tag{53}$$

The constant b_2^L in Eq. (52) can be calculated by

$$b_2^L = b_2(\xi_L) \tag{54}$$

with ξ_L being a large value of ξ. The region $|x| < a$ in Eq. (21) can be replaced by $0 \leqslant x < a$ in Eq. (51), because the temperature T is now an even function with respect to x. The integrand in Eq. (52) decays rapidly as ξ increases and the semi-infinite integral can be evaluated numerically using Filon's method. Therefore, Eq. (51) can now be solved for the coefficients c_n by the Schmidt method (Morse and Feshbach, 1958). The unknown A_1 is given by Eq. (49), and the entire temperature field can now be obtained.

Next, the stress field is found. It can be seen that the solutions of Eqs. (37) take the following forms for the layers i ($i = 1, 2, 3, 4$):

$$\begin{aligned}\bar{u}_i &= C_i\sinh(\alpha_{1i}y) + D_i\cosh(\alpha_{1i}y) + E_i\sinh(\alpha_{2i}y) + F_i\cosh(\alpha_{2i}y) + \mathrm{i}A_if_{2i}\sinh(|\xi|y/k_i)/\xi \\ &\quad + \mathrm{i}B_if_{2i}\cosh(|\xi|y/k_i)/\xi \\ \bar{v}_i &= \mathrm{i}\gamma_{1i}D_i\sinh(\alpha_{1i}y) + \mathrm{i}\gamma_{1i}C_i\cosh(\alpha_{1i}y) + \mathrm{i}\gamma_{2i}F_i\sinh(\alpha_{2i}y) + \mathrm{i}\gamma_{2i}E_i\cosh(\alpha_{2i}y) \\ &\quad + B_if_{3i}\sinh(|\xi|y/k_i)/|\xi| + A_if_{3i}\cosh(|\xi|y/k_i)/|\xi|\end{aligned} \tag{55}$$

where C_i, D_i, E_i, F_i are unknown coefficients and α_1, α_2 are the roots of the equation

$$\zeta_1\alpha^4 + \zeta_2\alpha^2 + \zeta_3 = 0. \tag{56}$$

In Eq. (55), γ_{1i}, γ_{2i}, f_{2i}, f_{3i} are expressed by

$$\gamma_{1i} = (\xi^2Q_{11i} - Q_{66i}\alpha_{1i}^2)/(L_i\xi\alpha_{1i}), \quad \gamma_{2i} = (\xi^2Q_{11i} - Q_{66i}\alpha_{2i}^2)/(L_i\xi\alpha_{2i}) \tag{57}$$

$$\begin{aligned}f_{2i} &= [\beta_{1i}(Q_{66i}k_i^2 - Q_{22i}) + \beta_{2i}L_i]k_i^2/f_{4i} \\ f_{3i} &= [Q_{66i}\beta_{2i}k_i + k_i^3(\beta_{1i}L_i - Q_{11i}\beta_{2i})]/f_{4i}\end{aligned} \tag{58}$$

with

$$f_{4i} = Q_{66i}Q_{22i} - k_i^2(Q_{66i}^2 + Q_{11i}Q_{22i} - L_i^2) + Q_{11i}Q_{66i}k_i^4. \tag{59}$$

For the upper half-plane (5) and the lower half-plane (6), the solutions of Eq. (37) have the forms

$$\begin{aligned}\bar{u}_5 &= C_5\exp(-\alpha_{15}y) + E_5\exp(-\alpha_{25}y) + (\mathrm{i}A_5f_{15}/\xi)\exp(-|\xi|y/k_5) \\ \bar{v}_5 &= -\mathrm{i}\gamma_{25}C_5\exp(-\alpha_{15}y) - \mathrm{i}\gamma_{25}E_5\exp(-\alpha_{25}y) - (A_5f_{25}/|\xi|)\exp(-|\xi|y/k_5)\end{aligned} \tag{60}$$

$$\begin{aligned}\bar{u}_6 &= C_6\exp(\alpha_{16}y) + E_6\exp(\alpha_{26}y) + (\mathrm{i}A_6f_{16}/\xi)\exp(|\xi|y/k_6) \\ \bar{v}_6 &= \mathrm{i}\gamma_{26}C_6\exp(\alpha_{16}y) + \mathrm{i}\gamma_{26}E_6\exp(\alpha_{26}y) + (A_6f_{26}/|\xi|)\exp(|\xi|y/k_6)\end{aligned} \tag{61}$$

where C_5, E_5, C_6, E_6 are unknown coefficients. The roots α_{1i} and α_{2i} ($i = 5, 6$) are chosen so as to have positive real parts due to the fact that the displacements $\bar{u}_5$, $\bar{v}_5$, $\bar{u}_6$, $\bar{v}_6$ vanish as y approaches $+\infty$ or $-\infty$.

Substituting Eqs. (55), (60), (61) into Eq. (39), stresses can be expressed in the Fourier domain. Using Eqs. (27), (30)–(33), the unknowns.

$E_1, F_1, C_2, D_2, E_2, F_2, C_3, D_3, E_3, F_3, C_4, D_4, E_4, F_4, C_5, E_5, C_6, E_6$ can be represented by the unknowns C_1, D_1 and a known A_1 as follows:

$$\begin{aligned}
&E_1 = C_1 g_{11} + D_1 g_{12} + \mathrm{i}A_1 g_{13}, \quad F_1 = C_1 g_{21} + D_1 g_{22} + \mathrm{i}A_1 g_{23}\\
&C_2 = C_1 g_{31} + D_1 g_{32} + \mathrm{i}A_1 g_{33}, \quad D_2 = C_1 g_{41} + D_1 g_{42} + \mathrm{i}A_1 g_{43}\\
&E_2 = C_1 g_{51} + D_1 g_{52} + \mathrm{i}A_1 g_{53}, \quad F_2 = C_1 g_{61} + D_1 g_{62} + \mathrm{i}A_1 g_{63}\\
&C_3 = C_1 g_{71} + D_1 g_{72} + \mathrm{i}A_1 g_{73}, \quad D_3 = C_1 g_{81} + D_1 g_{82} + \mathrm{i}A_1 g_{83}\\
&E_3 = C_1 g_{91} + D_1 g_{92} + \mathrm{i}A_1 g_{93}, \quad F_3 = C_1 g_{101} + D_1 g_{102} + \mathrm{i}A_1 g_{103}\\
&C_4 = C_1 g_{111} + D_1 g_{112} + \mathrm{i}A_1 g_{113}, \quad D_4 = C_1 g_{121} + D_1 g_{122} + \mathrm{i}A_1 g_{123}\\
&E_4 = C_1 g_{131} + D_1 g_{132} + \mathrm{i}A_1 g_{133}, \quad F_4 = C_1 g_{141} + D_1 g_{142} + \mathrm{i}A_1 g_{143}\\
&C_5 = C_1 g_{151} + D_1 g_{152} + \mathrm{i}A_1 g_{153}, \quad E_5 = C_1 g_{161} + D_1 g_{162} + \mathrm{i}A_1 g_{163}\\
&C_6 = C_1 g_{171} + D_1 g_{172} + \mathrm{i}A_1 g_{173}, \quad E_6 = C_1 g_{181} + D_1 g_{182} + \mathrm{i}A_1 g_{183}
\end{aligned} \tag{62}$$

where the expressions of the known functions $g_{11}, g_{12}, \ldots\ldots, g_{183}$, are shown in Appendix B.

The differences in the displacements at $y = 0$ can now be expressed in the Fourier domain by the equations

$$\bar{u}_1^0 - \bar{u}_2^0 = C_1 l_1 + D_1 l_2 + \mathrm{i}A_1 l_3, \quad \bar{v}_1^0 - \bar{v}_2^0 = \mathrm{i}C_1 l_4 + \mathrm{i}D_1 l_5 + A_1 l_6 \tag{63}$$

with

$$\begin{aligned}
&l_1 = g_{21} - g_{41} - g_{61}, \quad l_2 = 1 + g_{22} - g_{42} - g_{62}\\
&l_3 = g_{23} - g_{43} - g_{63} + [f_1(\xi) f_{11} - f_3(\xi) f_{12}]/\xi\\
&l_4 = \gamma_{11} + g_{11}\gamma_{21} - g_{31}\gamma_{12} - g_{51}\gamma_{22}, \quad l_5 = g_{12}\gamma_{21} - g_{32}\gamma_{12} - g_{52}\gamma_{22}\\
&l_6 = -g_{13}\gamma_{21} + g_{33}\gamma_{12} + g_{53}\gamma_{22} + [f_{21} - f_2(\xi) f_{12}]/|\xi|.
\end{aligned} \tag{64}$$

Eq. (29) shows that the displacements are continuous outside of the crack. To satisfy these conditions, the differences in the displacements are expanded by the series

$$\begin{aligned}
\pi(u_1^0 - u_2^0) &= \textstyle\sum_{n=1}^{\infty} d_n \sin[2n \sin^{-1}(x/a)] && \text{for } y = 0,\ |x| < a\\
&= 0 && \text{for } y = 0,\ a < |x|\\
\pi(v_1^0 - v_2^0) &= \textstyle\sum_{n=1}^{\infty} e_n \cos[(2n-1) \sin^{-1}(x/a)] && \text{for } y = 0,\ |x| < a\\
&= 0 && \text{for } y = 0,\ a < |x|
\end{aligned} \tag{65}$$

where d_n, e_n are the unknown coefficients to be determined. The Fourier-transformed expressions of Eq. (65) are

$$\begin{aligned}
(\bar{u}_1^0 - \bar{u}_2^0) &= \mathrm{i}\sum_{n=1}^{\infty} d_n (2n/\xi) J_{2n}(a\xi)\\
(\bar{v}_1^0 - \bar{v}_2^0) &= \sum_{n=1}^{\infty} e_n [(2n-1)/\xi] J_{2n-1}(a\xi).
\end{aligned} \tag{66}$$

Using Eqs. (63) and (66), the unknown functions C_1 and D_1 can be represented by the known function A_1 and the unknown coefficients d_n, e_n as follows:

$$
\begin{aligned}
C_1 &= \mathrm{i}\sum_{n=1}^{\infty} d_n(2n)l_5J_{2n}(a\xi)/(\xi\Delta'') + \mathrm{i}\sum_{n=1}^{\infty} e_n(2n-1)l_2J_{2n-1}(a\xi)/(\xi\Delta'') - \mathrm{i}A_1(l_3l_5 + l_6l_2)/\Delta'' \\
D_1 &= -\mathrm{i}\sum_{n=1}^{\infty} d_n(2n)l_4J_{2n}(a\xi)/(\xi\Delta'') - \mathrm{i}\sum_{n=1}^{\infty} e_n(2n-1)l_1J_{2n-1}(a\xi)/(\xi\Delta'') + \mathrm{i}A_1(l_3l_4 + l_6l_1)/\Delta''
\end{aligned} \tag{67}
$$

with

$$\Delta'' = l_1l_5 - l_2l_4. \tag{68}$$

Substituting Eq. (67) into the Fourier-transformed expressions for the stresses at $y = 0$, and by inverting them into the physical domain, we can obtain

$$
\begin{aligned}
\tau^0_{yy1} = {}& \sum_{n=1}^{\infty} d_n(2n)/\pi \int_0^{\infty} [Q_1(\xi)/\xi]J_{2n}(a\xi)\cos(\xi x)\,\mathrm{d}\xi \\
&+ \sum_{n=1}^{\infty} e_n(2n-1)/\pi \int_0^{\infty} [Q_2(\xi)/\xi]J_{2n-1}(a\xi)\cos(\xi x)\,\mathrm{d}\xi \\
&+ \sum_{n=1}^{\infty} c_n(2n-1)/\pi \int_0^{\infty} [Q_3(\xi)/\xi]J_{2n-1}(a\xi)\cos(\xi x)\,\mathrm{d}\xi \\
\tau^0_{xy1} = {}& \sum_{n=1}^{\infty} d_n(2n)/\pi \int_0^{\infty} [Q_4(\xi)/\xi]J_{2n}(a\xi)\sin(\xi x)\,\mathrm{d}\xi \\
&+ \sum_{n=1}^{\infty} e_n(2n-1)/\pi \int_0^{\infty} [Q_5(\xi)/\xi]J_{2n-1}(a\xi)\sin(\xi x)\,\mathrm{d}\xi \\
&+ \sum_{n=1}^{\infty} c_n(2n-1)/\pi \int_0^{\infty} [Q_6(\xi)/\xi]J_{2n-1}(a\xi)\sin(\xi x)\,\mathrm{d}\xi
\end{aligned} \tag{69}
$$

where the expressions of the known functions $Q_1(\xi), Q_2(\xi), \ldots, Q_6(\xi)$ are shown in Appendix C.

If the functions $Q_1(\xi), Q_2(\xi), \ldots, Q_6(\xi)$ are calculated numerically, it can be seen that the $Q_i(\xi)$ decay rapidly as ξ increases for $i = 1, 3, 5$. The behavior of the functions $Q_i(\xi)$ as ξ increases is given by

$$Q_i(\xi)/\xi \to Q_i^L \quad (i = 2, 4), \quad Q_i(\xi) \to Q_i^0 \quad (i = 6) \tag{70}$$

where the constants Q_i^L and Q_i^0 can be calculated as

$$Q_i^L = Q_i(\xi_L)/\xi_L \quad (i = 2, 4), \quad Q_i^0 = Q_i(\xi_L) \quad (i = 6) \tag{71}$$

with ξ_L being a large value of ξ.

Finally, the remaining boundary condition (28) can be reduced to the form

$$
\begin{aligned}
&\sum_{n=1}^{\infty} d_nG_n(x) + \sum_{n=1}^{\infty} e_nH_n(x) = -U(x) \\
&\sum_{n=1}^{\infty} d_nK_n(x) + \sum_{n=1}^{\infty} e_nL_n(x) = -V(x) \quad \text{for } 0 \leqslant x < a
\end{aligned} \tag{72}
$$

with

$$G_n(x) = (2n)/\pi \int_0^\infty [Q_1(\xi)/\xi] J_{2n}(a\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$H_n(x) = (2n-1)/\pi \left\{ \int_0^\infty [Q_2(\xi)/\xi - Q_2^L] J_{2n-1}(a\xi)\cos(\xi x)\,\mathrm{d}\xi + Q_2^L \cos[(2n-1)\sin^{-1}(x/a)]/(a^2-x^2)^{1/2} \right\}$$

$$K_n(x) = (2n)/\pi \left\{ \int_0^\infty [Q_4(\xi)/\xi - Q_4^L] J_{2n}(a\xi)\sin(\xi x)\,\mathrm{d}\xi + Q_4^L \sin[2n\sin^{-1}(x/a)]/(a^2-x^2)^{1/2} \right\}$$

$$L_n(x) = (2n-1)/\pi \int_0^\infty [Q_5(\xi)/\xi] J_{2n-1}(a\xi)\sin(\xi x)\,\mathrm{d}\xi \tag{73}$$

$$U(x) = \sum_{n=1}^{\infty} c_n(2n-1)/(2\pi) \int_0^\infty [Q_3(\xi)/\xi] J_{2n-1}(a\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$V(x) = \sum_{n=1}^{\infty} c_n(2n-1)/(2\pi) \left\{ \int_0^\infty [Q_6(\xi)/\xi - Q_6^0/\xi] J_{2n-1}(a\xi)\sin(\xi x)\,\mathrm{d}\xi + Q_6^0 \sin[(2n-1)\sin^{-1}(x/a)]/(2n-1) \right\}. \tag{74}$$

Eq. (72) can now be solved for a determination of the coefficients d_n, e_n by using the Schmidt method (Yau, 1967).

Since the coefficients c_n, d_n, e_n are now known, the entire temperature and stress fields can be obtained. The stresses τ^0_{yy1}, τ^0_{xy1} at $y = 0$ are shown by Eq. (69). If we slightly modify the integrands in Eq. (69), and by using the relations

$$\int_0^\infty J_n(a\xi)[\cos(\xi x), \sin(\xi x)]\,\mathrm{d}\xi = \Big\{ -a^n(x^2-a^2)^{-1/2}[x+(x^2-a^2)^{-1/2}]^{-n}\sin(n\pi/2),$$
$$a^n(x^2-a^2)^{-1/2}[x+(x^2-a^2)^{-1/2}]^{-n}\cos(n\pi/2) \Big\} \quad \text{for } a < x \tag{75}$$

the stress intensity factors can be determined as follows:

$$K_1 = [2\pi(x-a)]^{1/2}\tau^0_{yy1}|_{x\to a+} = \sum_{n=1}^{\infty} e_n(2n-1)(-1)^{n-1}Q_2^L/(\pi a)^{1/2}$$
$$K_2 = [2\pi(x-a)]^{1/2}\tau^0_{xy1}|_{x\to a+} = \sum_{n=1}^{\infty} d_n(2n)(-1)^{n}Q_4^L/(\pi a)^{1/2}. \tag{76}$$

The analysis presented in the Sections 3 and 4 is that for a case in which the nonhomogeneous interfacial layer has been replaced by three homogeneous layers. Namely, the stress intensity factors were solved only for the case $m = 3$. The solutions for $m = 5, 7$ are quite straightforward.

The values K_1 and K_2 are calculated numerically for $m = 3, 5, 7$ and are plotted with respect to $1/m$. The nonhomogeneous interfacial layer (A) can be replaced by an infinite number of homogeneous layers. Then, the results for the interfacial layer, in which the material properties are assumed to vary continuously with respect to y, can be obtained as the value of $m \to \infty$, or namely as $1/m \to 0$. This process is explained in detail below.

A polar coordinate system (r, θ) is described as shown in Fig. 5. Rectangular co-ordinates (x, y) are related to polar co-ordinates (r, θ) by the equation

$$x = a + r\cos\theta, \quad y = r\sin\theta. \tag{77}$$

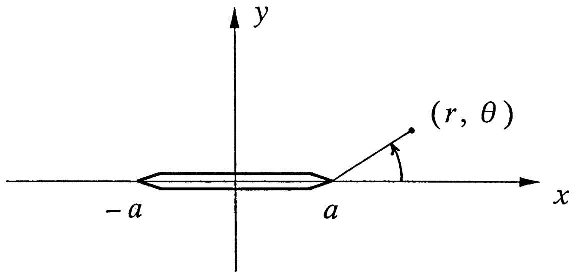

Fig. 5. Polar co-ordinates with origin at crack end.

Let us define the stress intensity factor $K_{\theta\theta}$ by the definition

$$K_{\theta\theta} = \lim_{r\to 0} \sqrt{2\pi r}\tau_{\theta\theta}. \tag{78}$$

Then, $K_{\theta\theta}$ can be expressed in terms of the stress intensity factors K_1 and K_2 as follows (Williams, 1957):

$$K_{\theta\theta} = K_1 \cos^3(\theta/2) - 3K_2 \cos(\theta/2)\sin(\theta)/2. \tag{79}$$

5. Numerical examples and results

If a diffusion method is used to join a cramics and a metal, a thin diffusion layer appears between these materials (Iwamoto and Soumiya, 1990). It is likely that the material properties in the layer vary linearly. In the numerical calculations, it is considered that a ceramics plate is joined with a steel plate by a diffusion method. Namely, half-plane (C), in Fig. 1, is a ceramic plate and half-plane (B) is a steel plate. The present analysis is based on the orthotropic elasticity. This presents no problem when solving the temperature field for an isotropic material. However, Eq. (56) has two kinds of multiple roots. Namely, α_{1i} is equal to α_{2i}, and then the solutions given according to the Eqs. (55), (60) and (61) are invalid. If the value of v_{yx} is replaced by a value slightly larger than v_{xy}, the expressions given in Eqs. (55), (60) and (61) are still valid. The elastic constants used in the numerical calculations are listed in Table 1. The semi-infinite integral in Eq. (52) and those in Eqs. (73) and (74) can be easily evaluated numerically using Filon's method because the integrands decay rapidly as the integration variable ξ increases.

The Schmidt method has been applied to obtain the coefficients c_n in Eq. (51) and d_n, e_n in Eq. (72), truncating the infinite series to nine terms. It has been verified that the left-hand side of Eq. (51) coincides

Table 1
Elastic constants

Constants	Steel	Ceramics (Si_3N_4)
E_{xx} (GPa)	205.9	304.8
μ_{xy} (GPa)	79.2	120.0
v_{xy}	0.3	0.27
v_{yx}	0.3×1.01	0.27×1.01
α_{xx} ($\times 10^{-5}$/°C)	1.14	0.29
α_{yy} ($\times 10^{-5}$/°C)	1.14	0.29
k_x [W/(m°C)]	48.6	15.5
k_y [W/(m°C)]	48.6	15.5

with the right hand side of Eq. (51). The same applies to Eq. (72). Namely, it can been seen that the boundary conditions inside of the crack are completely satisfied.

It is assumed that the crack is situated on the mid-surface of the interfacial layer. Namely, the H_B/H_C ratio is set to 1.0. In principle, the interfacial layer (A) with a thickness of $(H_B + H_C)$ is divided into m layers, although not necessarily of equal thickness. In the solution presented here, the layer (A) is assumed to be divided into layers of equal thickness $(H_B + H_C)/m$.

The stress intensity factor K_1 is calculated for $m = 3, 5, 7$ and $H_B/a = 0.1$. Using the results for $m = 3, 5, 7$, K_1 is approximated by the relation:

$$K_1/\{E_{xx1}\alpha_{xx1}\sqrt{\pi}a^{1.5}t/4\} = a_1(1/m)^3 + a_2(1/m)^2 + a_3 \tag{80}$$

where constants a_1, a_2 and a_3 can be easily determined. The stress intensity factor K_2 is also obtained in this manner. The results of K_1 and K_2 are plotted with respect to $1/m$ in Fig. 6. Values for $m \geqslant 9$ are not calculated numerically because these values can be inferred, and the curves for $1/m < 1/7$ are shown by the broken lines. The material properties are thought to vary continuously across the interfacial layer. This layer can be replaced by an infinite number of infinitesimally thin layers. Therefore, the value for the interfacial layer can be given by the values of the curves at $1/m \to 0$ in Fig. 6. Namely, a constant a_3 remains important and it is the correct value of $K_1/\{E_{xx1}\alpha_{xx1}\sqrt{\pi}a^{1.5}t/4\}$.

For $H_B/a = 0.1$, 0.2, 0.3, 0.4, 0.8, the stress intensity factors K_1 and K_2 are obtained in the same manner described above and these are plotted with respect to H_B/a in Fig. 7. The Schmidt method cannot be applied with sufficient accuracy to $H_B/a < 0.1$.

By experimental analysis, Erdogan and Sih (1963) verified that crack extension in brittle materials initiates in a plane perpendicular to the direction of the greatest tension. Thereby, the values $K_{\theta\theta}$ are calculated by substituting K_1 and K_2 into Eq. (79) for $H_B/a = 0.1$, 0.4, 0.8, and these are plotted versus θ in Fig. 8.

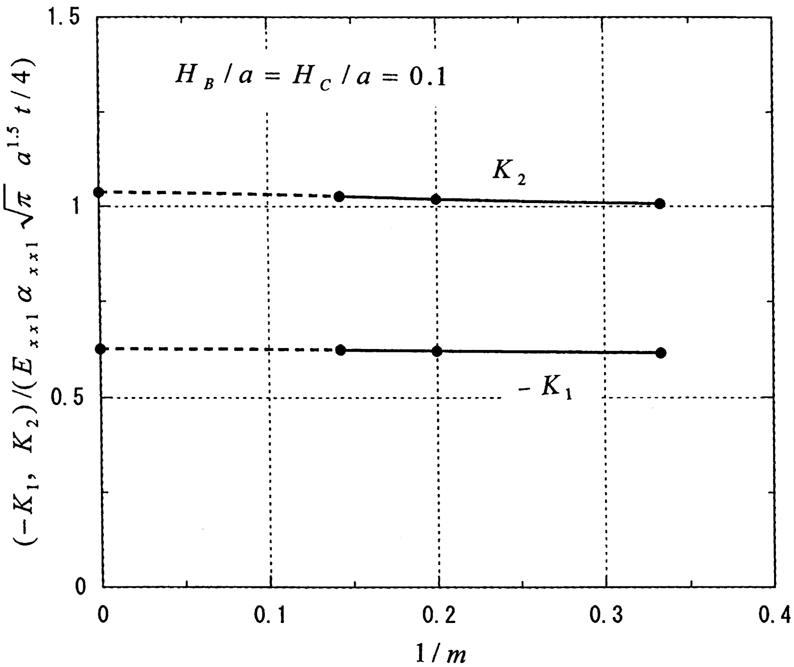

Fig. 6. Stress intensity factors K_1 and K_2 for $H_B/a = 0.1$ versus $1/m$.

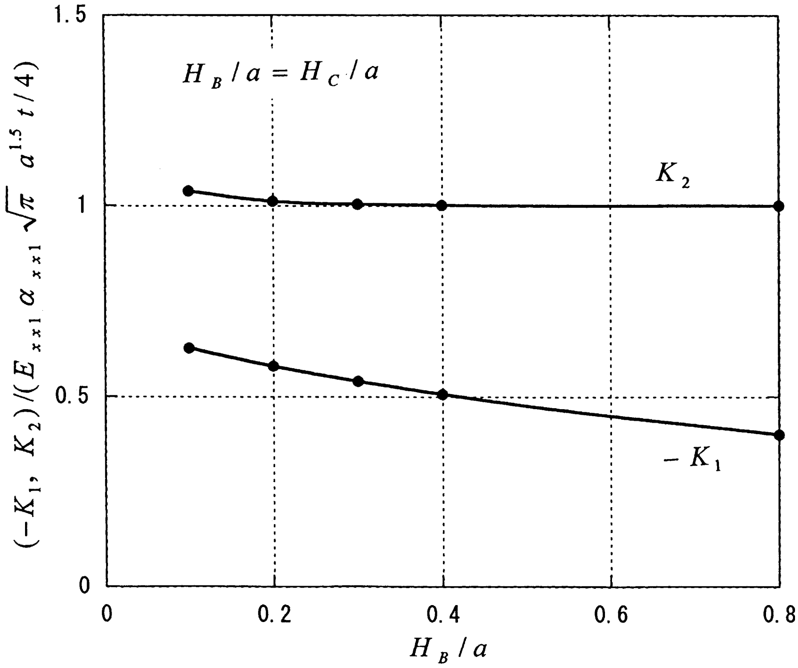

Fig. 7. Stress intensity factors K_1 and K_2 for ceramics–steel composites versus H_B/a.

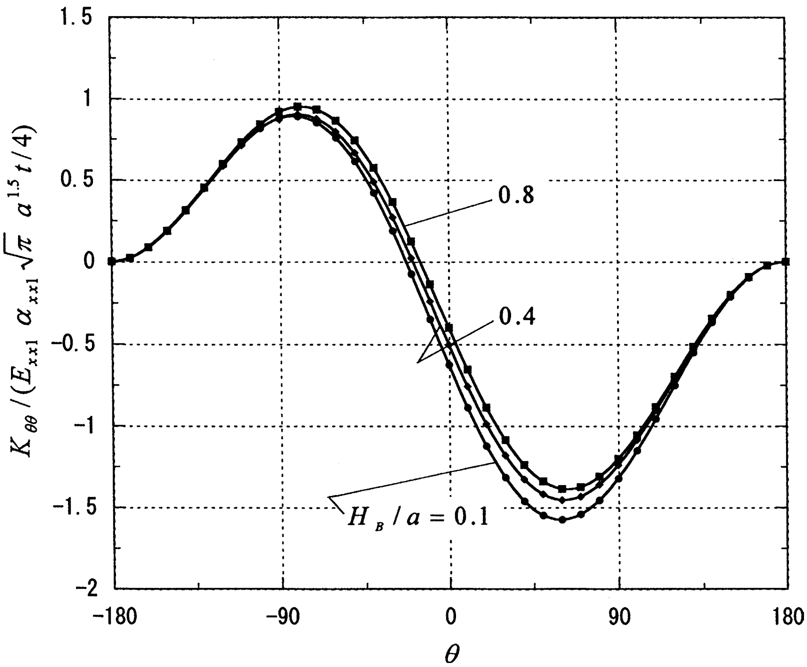

Fig. 8. Stress intensity factor $K_{\theta\theta}$ versus θ [°] for $H_B/a(=H_C/a) = 0.8, 0.4, 0.1$.

6. Discussion

If a ceramics is joined with a steel by using a diffusion method, a thin diffusion layer appears between these materials. Consider that the composites are used in high temperature environments. In this case, the value of the stress intensity factor K_1 around an interface crack is negative, as shown in Fig. 7. This means that τ_{yy} near the crack tip is negative. In linear fracture mechanics, the shape of a crack is assumed to be a thin ellipse. Then, the crack faces may approach each other but these do not come into contact because it is assumed that the thickness of the crack is enough to avoid the contact.

It is considered that the diffusion layer is brittle. The maximum value of the stress intensity factor $K_{\theta\theta}$ occurs at $\theta = -85°$. Then, it is very likely that the crack may extend at this direction when the value of $K_{\theta\theta}$ reaches the fracture toughness value.

Appendix A

$$f_i(\xi) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{19} \\ a_{21} & a_{22} & \cdots & a_{2i-1} & b_2 & a_{2i+1} & \cdots & a_{29} \\ \vdots & & & & & & \vdots & \\ a_{81} & a_{82} & \cdots & a_{8i-1} & b_8 & a_{8i+1} & \cdots & a_{89} \\ a_{91} & a_{92} & \cdots & a_{9i-1} & b_9 & a_{9i+1} & \cdots & a_{99} \end{vmatrix} \Big/ \Delta \quad (i = 1, 2, \ldots, 8, 9) \tag{A.1}$$

with

$$\Delta = |a_{ij}| \quad (i, j = 1, 2, \ldots, 8, 9) \tag{A.2}$$

and

$$\begin{aligned}
&a_{11} = 0, \quad a_{12} = 1.0, \quad a_{13} = 0, \quad a_{14} = 0, \quad a_{15} = 0, \quad a_{16} = 0, \quad a_{17} = 0, \quad a_{18} = 0, \quad a_{19} = 0 \quad b_1 = 1.0 \\
&a_{21} = -\sinh(|\xi|H_1/k_1), \quad a_{22} = 0.0, \quad a_{23} = 0, \quad a_{24} = \cosh(|\xi|H_1/k_3) \\
&a_{25} = \sinh(|\xi|H_1/k_3), \quad a_{26} = 0, \quad a_{27} = 0, \quad a_{28} = 0, \quad a_{29} = 0, \quad b_2 = \cosh(|\xi|H_1/k_1) \\
&a_{31} = -\cosh(|\xi|H_1/k_1), \quad a_{32} = 0.0, \quad a_{33} = 0, \quad a_{34} \sin sh(|\xi|H_1/k_3) \\
&a_{35} = \cosh(|\xi|H_1/k_3), \quad a_{36} = 0, \quad a_{37} = 0, \quad a_{38} = 0, \quad a_{39} = 0, \quad b_3 = \sinh(|\xi|H_1/k_1) \\
&a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = 0, \quad a_{44} = -\cosh(|\xi|H_3/k_3), \quad a_{45} = -\sinh(|\xi|H_3/k_3) \\
&a_{46} = 0, \quad a_{47} = 0, \quad a_{48} = -\exp(-|\xi|H_3/k_5), \quad a_{49} = 0, \quad b_4 = 0 \\
&a_{51} = 0, \quad a_{52} = 0, \quad a_{53} = 0, \quad a_{54} = -\sinh(|\xi|H_3/k_3), \quad a_{55} = -\cosh(|\xi|H_3/k_3) \\
&a_{56} = 0, \quad a_{57} = 0, \quad a_{58} = \exp(-|\xi|H_3/k_5), \quad a_{59} = 0, \quad b_5 = 0 \\
&a_{61} = 0, \quad a_{62} = \cosh(-|\xi|H_2/k_2), \quad a_{63} = \sinh(-|\xi|H_2/k_2), \quad a_{64} = 0, \quad a_{65} = 0 \\
&a_{66} = -\cosh(-|\xi|H_2/k_4), \quad a_{67} = -\sinh(-|\xi|H_2/k_4), \quad a_{68} = 0, \quad a_{69} = 0, \quad b_6 = 0 \\
&a_{71} = 0, \quad a_{72} = \sinh(-|\xi|H_2/k_2), \quad a_{73} = \cosh(-|\xi|H_2/k_2), \quad a_{74} = 0, \quad a_{75} = 0 \\
&a_{76} = -\sinh(-|\xi|H_2/k_4), \quad a_{77} = -\cosh(-|\xi|H_2/k_4), \quad a_{78} = 0, \quad a_{79} = 0, \quad b_7 = 0 \\
&a_{81} = 0, \quad a_{82} = 0, \quad a_{83} = 0, \quad a_{84} = 0, \quad a_{85} = 0, \quad a_{86} = \cosh(-|\xi|H_4/k_4) \\
&a_{87} = \sinh(-|\xi|H_4/k_4), \quad a_{88} = 0, \quad a_{89} = -\exp(-|\xi|H_4/k_6), \quad b_8 = 0 \\
&a_{91} = 0, \quad a_{92} = 0, \quad a_{93} = 0, \quad a_{94} = 0, \quad a_{95} = 0, \quad a_{96} = \sinh(-|\xi|H_4/k_4) \\
&a_{97} = \cosh(-|\xi|H_4/k_4), \quad a_{98} = 0, \quad a_{99} = -\exp(-|\xi|H_4/k_6), \quad b_9 = 0
\end{aligned} \tag{A.3}$$

Appendix B

$$g_{ij} = \left| \begin{array}{cccccccc} a'_{11} & a'_{12} & \cdots & a'_{1i-1} & b'_{1j} & a'_{1i+1} & \cdots & a'_{118} \\ a'_{21} & a'_{22} & \cdots & a'_{2i-1} & b'_{2j} & a'_{2i+1} & \cdots & a'_{218} \\ \vdots & & & & & & & \vdots \\ a'_{171} & a'_{172} & \cdots & a'_{17i-1} & b'_{17j} & a'_{17i+1} & \cdots & a'_{1718} \\ a'_{181} & a'_{182} & \cdots & a'_{18i-1} & b'_{18j} & a'_{18i+1} & \cdots & a'_{1818} \end{array} \right| \Bigg/ \Delta' \quad (i = 1, 2, \ldots, 17, 18)\ (j = 1, 2, 3) \tag{B.1}$$

with

$$\Delta' = \left| a'_{kl} \right| \quad (k, l = 1, 2, \ldots, 17, 18) \tag{B.2}$$

and

$$a'_{11} = 0, \quad a'_{12} = -Q_{121}\xi + Q_{221}\gamma_{21}\alpha_{21}, \quad a'_{13} = 0, \quad a'_{14} = Q_{122}\xi - Q_{222}\gamma_{12}\alpha_{12}$$

$$a'_{15} = 0, \quad a'_{16} = Q_{122}\xi - Q_{222}\gamma_{22}\alpha_{22}, \quad a'_{17} = 0, \quad a'_{18} = 0, \quad a'_{19} = 0, \quad a'_{110} = 0$$

$$a'_{111} = 0, \quad a'_{112} = 0, \quad a'_{113} = 0, \quad a'_{114} = 0, \quad a'_{115} = 0, \quad a'_{116} = 0, \quad a'_{117} = 0$$

$$a'_{118} = 0, \quad b'_{11} = 0, \quad b'_{12} = Q_{121}\xi - Q_{221}\gamma_{11}\alpha_{11}$$

$$b'_{13} = f_1(\xi)[Q_{121}f_{11} + Q_{221}f_{21}/k_1 - \beta_{21}] - f_3(\xi)[Q_{122}f_{12} + Q_{222}f_{22}/k_2 - \beta_{22}]$$

$$a'_{21} = Q_{661}(\alpha_{21} + \xi\gamma_{21}), \quad a'_{22} = 0, \quad a'_{23} = -Q_{662}(\alpha_{12} + \xi\gamma_{12}), \quad a'_{24} = 0$$

$$a'_{25} = -Q_{662}(\alpha_{22} + \xi\gamma_{22}), \quad a'_{26} = 0, \quad a'_{27} = 0, \quad a'_{28} = 0, \quad a'_{29} = 0, \quad a'_{210} = 0$$

$$a'_{211} = 0, \quad a'_{212} = 0, \quad a'_{213} = 0, \quad a'_{214} = 0, \quad a'_{215} = 0, \quad a'_{216} = 0, \quad a'_{217} = 0$$

$$b'_{218} = 0, \quad b'_{21} = -Q_{661}(\alpha_{11} + \xi\gamma_{11}), \quad b'_{22} = 0$$

$$b'_{23} = -[Q_{661}(f_{11}/k_1 - f_{21}) + f_2(\xi)Q_{662}(f_{12}/k_2 - f_{22})]|\xi|/\xi$$

$$a'_{31} = -[-Q_{121}\xi + Q_{221}\gamma_{21}\alpha_{21}]\sinh(\alpha_{21}H_1), \quad a'_{32} = -[-Q_{121}\xi + Q_{221}\gamma_{21}\alpha_{21}]\cosh(\alpha_{21}H_1)$$

$$a'_{33} = 0, \quad a'_{34} = 0, \quad a'_{35} = 0, \quad a'_{36} = 0, \quad a'_{37} = [-Q_{123}\xi + Q_{223}\gamma_{13}\alpha_{13}]\sinh(\alpha_{13}H_1)$$

$$a'_{38} = [-Q_{123}\xi + Q_{223}\gamma_{13}\alpha_{13}]\cosh(\alpha_{13}H_1), \quad a'_{39} = [-Q_{123}\xi + Q_{223}\gamma_{23}\alpha_{23}]\sinh(\alpha_{23}H_1)$$

$$a'_{310} = [-Q_{123}\xi + Q_{223}\gamma_{23}\alpha_{23}]\cosh(\alpha_{23}H_1), \quad a'_{311} = 0, \quad a'_{312} = 0, \quad a'_{313} = 0, \quad a'_{314} = 0$$

$$a'_{315} = 0, \quad a'_{316} = 0, \quad a'_{317} = 0, \quad a'_{318} = 0$$

$$b'_{31} = [-Q_{121}\xi + Q_{221}\gamma_{11}\alpha_{11}]\sinh(\alpha_{11}H_1), \quad b'_{32} = [-Q_{121}\xi + Q_{221}\gamma_{11}\alpha_{11}]\cosh(\alpha_{11}H_1)$$

$$b'_{33} = [Q_{123}f_{13} + Q_{223}f_{23}/k_3 - \beta_{23}][f_4(\xi)\sinh(|\xi|H_1/k_3) + f_5(\xi)\cosh(|\xi|H_1/k_3)] \\ - [Q_{121}f_{11} + Q_{221}f_{21}/k_1 - \beta_{13}][\sinh(|\xi|H_1/k_1) + f_1(\xi)\cosh(|\xi|H_1/k_1)]$$

$$a'_{41} = Q_{661}(\alpha_{21} + \xi\gamma_{21})\cosh(\alpha_{21}H_1), \quad a'_{42} = Q_{661}(\alpha_{21} + \xi\gamma_{21})\sinh(\alpha_{21}H_1)$$

$$a'_{43} = 0, \quad a'_{44} = 0, \quad a'_{45} = 0, \quad a'_{46} = 0, \quad a'_{47} = Q_{663}(\alpha_{13} + \xi\gamma_{13})\cosh(\alpha_{13}H_1)$$

$$a'_{48} = Q_{663}(\alpha_{13} + \xi\gamma_{13})\sinh(\alpha_{13}H_1), \quad a'_{49} = Q_{663}(\alpha_{23} + \xi\gamma_{23})\cosh(\alpha_{23}H_1)$$

$$a'_{410} = Q_{663}(\alpha_{23} + \xi\gamma_{23})\sinh(\alpha_{23}H_1), \quad a'_{411} = 0, \quad a'_{412} = 0, \quad a'_{413} = 0, \quad a'_{414} = 0$$

$$a'_{415} = 0, \quad a'_{416} = 0, \quad a'_{417} = 0, \quad b'_{418} = 0, \quad b'_{41} = Q_{661}(\alpha_{11} + \xi\gamma_{11})\cosh(\alpha_{11}H_1)$$

$$b'_{42} = Q_{661}(\alpha_{11} + \xi\gamma_{11})\sinh(\alpha_{11}H_1)$$

$$b'_{43} = -Q_{663}(f_{13}/k_3 - f_{23})(|\xi|/\xi)[f_4(\xi)\cosh(|\xi|H_1/k_3) + f_5(\xi)\sinh(|\xi|H_1/k_3) \\ + Q_{661}(f_{11}/k_1 - f_{21})(|\xi|/\xi)[\cosh(|\xi|H_1/k_1) + f_1(\xi)\sinh(|\xi|H_1/k_1)]$$

$$a'_{51} = -\sinh(\alpha_{21}H_1), \quad a'_{52} = -\cosh(\alpha_{21}H_1), \quad a'_{53} = 0, \quad a'_{54} = 0, \quad a'_{55} = 0, \quad a'_{56} = 0$$

$$a'_{57} = \sinh(\alpha_{13}H_1), \quad a'_{58} = \cosh(\alpha_{13}H_1), \quad a'_{59} = \sinh(\alpha_{23}H_1), \quad a'_{510} = \cosh(\alpha_{23}H_1)$$

$$a'_{511} = 0, \quad a'_{512} = 0, \quad a'_{513} = 0, \quad a'_{514} = 0, \quad a'_{515} = 0, \quad a'_{516} = 0, \quad a'_{517} = 0$$

$$a'_{518} = 0, \quad b'_{51} = \sinh(\alpha_{11}H_1), \quad b'_{52} = \cosh(\alpha_{11}H_1)$$

$$b'_{53} = -[f_4(\xi)\sinh(|\xi|H_1/k_3) + f_5(\xi)\cosh(|\xi|H_1/k_3)]f_{13}/\xi \\ + [\sinh(|\xi|H_1/k_1) + f_1(\xi)\cosh(|\xi|H_1/k_1)]f_{11}/\xi$$

$$a'_{61} = -\gamma_{21}\cosh(\alpha_{21}H_1), \quad a'_{62} = -\gamma_{21}\sinh(\alpha_{21}H_1), \quad a'_{63} = 0, \quad a'_{64} = 0$$

$$a'_{65} = 0, \quad a'_{66} = 0, \quad a'_{67} = \gamma_{13}\cosh(\alpha_{13}H_1), \quad a'_{68} = \gamma_{13}\sinh(\alpha_{13}H_1)$$

$$a'_{69} = \gamma_{23}\cosh(\alpha_{23}H_1), \quad a'_{610} = \gamma_{23}\sinh(\alpha_{23}H_1), \quad a'_{611} = 0, \quad a'_{612} = 0$$

$$a'_{613} = 0, \quad a'_{614} = 0, \quad a'_{615} = 0, \quad a'_{616} = 0, \quad a'_{617} = 0, \quad a'_{618} = 0$$

$$b'_{61} = \gamma_{11}\cosh(\alpha_{11}H_1), \quad b'_{62} = \gamma_{11}\sinh(\alpha_{11}H_1)$$

$$b'_{63} = [f_5(\xi)\sinh(|\xi|H_1/k_3) + f_4(\xi)\cosh(|\xi|H_1/k_3)]f_{23}/|\xi| \\ - [f_1(\xi)\sinh(|\xi|H_1/k_1) + \cosh(|\xi|H_1/k_1)]f_{22}/|\xi|$$

$$a'_{71} = 0, \quad a'_{72} = 0, \quad a'_{73} = 0, \quad a'_{74} = 0, \quad a'_{75} = 0, \quad a'_{76} = 0$$

$$a'_{77} = -(-\xi Q_{123} + \gamma_{13}\alpha_{13}Q_{223})\sinh(\alpha_{13}H_3), \quad a'_{78} = -(-\xi Q_{123} + \gamma_{13}\alpha_{13}Q_{223})\cosh(\alpha_{13}H_3)$$

$$a'_{79} = -(-\xi Q_{123} + \gamma_{23}\alpha_{23}Q_{223})\sinh(\alpha_{23}H_3)$$

$$a'_{710} = -(-\xi Q_{123} + \gamma_{23}\alpha_{23}Q_{223})\cosh(\alpha_{23}H_3), \quad a'_{711} = 0, \quad a'_{712} = 0, \quad a'_{713} = 0$$

$$a'_{714} = 0, \quad a'_{715} = (-\xi Q_{125} + \gamma_{15}\alpha_{15}Q_{225})\exp(-\alpha_{15}H_3)$$

$$a'_{716} = (-\xi Q_{125} + \gamma_{25}\alpha_{25}Q_{225})\exp(-\alpha_{25}H_3), \quad a'_{717} = 0, \quad a'_{718} = 0, \quad b'_{71} = 0, \quad b'_{72} = 0$$

$$b'_{73} = (Q_{125}f_{15} + Q_{225}f_{25}/k_5 - \beta_{25})f_8(\xi)\exp(-|\xi|H_3/k_5) - (Q_{123}f_{13} + Q_{223}f_{23}/k_3 - \beta_{23}) \times [f_4(\xi)\sinh(|\xi|H_3/k_3) + f_5(\xi)\cosh(|\xi|H_3/k_3)]$$

$$a'_{81} = 0, \quad a'_{82} = 0, \quad a'_{83} = 0, \quad a'_{84} = 0, \quad a'_{85} = 0, \quad a'_{86} = 0$$

$$a'_{87} = -Q_{663}(\alpha_{13} + \xi\gamma_{13})\cosh(\alpha_{13}H_3), \quad a'_{88} = -Q_{663}(\alpha_{13} + \xi\gamma_{13})\sinh(\alpha_{13}H_3)$$

$$a'_{89} = -Q_{663}(\alpha_{23} + \xi\gamma_{23})\cosh(\alpha_{23}H_3), \quad a'_{810} = -Q_{663}(\alpha_{23} + \xi\gamma_{23})\sinh(\alpha_{23}H_3)$$

$$a'_{811} = 0, \quad a'_{812} = 0, \quad a'_{813} = 0, \quad a'_{814} = 0, \quad a'_{815} = -Q_{665}(\alpha_{15} + \xi\gamma_{15})\exp(-\alpha_{15}H_3)$$

$$a'_{816} = -Q_{665}(\alpha_{25} + \xi\gamma_{25})\exp(-\alpha_{25}H_3), \quad a'_{817} = 0, \quad a'_{818} = 0, \quad b'_{81} = 0, \quad b'_{82} = 0$$

$$b'_{83} = Q_{665}(f_{15}/k_5 - f_{25})\exp(-|\xi|H_3/k_5)|\xi|/\xi + Q_{663}(f_{13}/k_3 - f_{23}) \times [f_4(\xi)\cosh(|\xi|H_3/k_3) + f_5(\xi)\sinh(|\xi|H_3/k_3)]|\xi|/\xi$$

$$a'_{91} = 0, \quad a'_{92} = 0, \quad a'_{93} = 0, \quad a'_{94} = 0, \quad a'_{95} = 0, \quad a'_{96} = 0, \quad a'_{97} = -\sinh(\alpha_{13}H_3)$$

$$a'_{98} = -\cosh(\alpha_{13}H_3), \quad a'_{99} = -\sinh(\alpha_{23}H_3), \quad a'_{910} = -\cosh(\alpha_{23}H_3), \quad a'_{911} = 0$$

$$a'_{912} = 0, \quad a'_{913} = 0, \quad a'_{914} = 0, \quad a'_{915} = \exp(-\alpha_{15}H_3), \quad a'_{916} = \exp(-\alpha_{25}H_3)$$

$$a'_{917} = 0, \quad a'_{918} = 0, \quad b'_{91} = 0, \quad b'_{92} = 0$$

$$b'_{93} = -f_8(\xi)f_{15}\exp(-|\xi|H_3/k_5)/\xi + [f_4(\xi)\sinh(|\xi|H_3/k_3) + f_5(\xi)\cosh(|\xi|H_3/k_3)]f_{13}/\xi$$

$$a'_{101} = 0, \quad a'_{102} = 0, \quad a'_{103} = 0, \quad a'_{104} = 0, \quad a'_{105} = 0, \quad a'_{106} = 0$$

$$a'_{107} = -\gamma_{13}\cosh(\alpha_{13}H_3), \quad a'_{108} = -\gamma_{13}\sinh(\alpha_{13}H_3), \quad a'_{109} = -\gamma_{23}\cosh(\alpha_{23}H_3)$$

$$a'_{1010} = -\gamma_{23}\sinh(\alpha_{23}H_3), \quad a'_{1011} = 0, \quad a'_{1012} = 0, \quad a'_{1013} = 0, \quad a'_{1014} = 0$$

$$a'_{1015} = -\gamma_{15}\exp(-\alpha_{15}H_3), \quad a'_{1016} = -\gamma_{25}\exp(-\alpha_{25}H_3), \quad a'_{1017} = 0, \quad a'_{1018} = 0$$

$$b'_{101} = 0, \quad b'_{102} = 0$$

$$b'_{103} = -f_8(\xi)\exp(-|\xi|H_3/k_5)f_{25}/|\xi| - [f_5(\xi)\sinh(|\xi|H_3/k_3) + f_4(\xi)\cosh(|\xi|H_3/k_3)]f_{23}/|\xi|$$

$$a'_{111} = 0, \quad a'_{112} = 0, \quad a'_{113} = (-Q_{122}\xi + Q_{222}\gamma_{12}\alpha_{12})\sinh(-\alpha_{12}H_2)$$

$$a'_{114} = (-Q_{122}\xi + Q_{222}\gamma_{12}\alpha_{12})\cosh(-\alpha_{12}H_2)$$

$$a'_{115} = (-Q_{122}\xi + Q_{222}\gamma_{22}\alpha_{22})\sinh(-\alpha_{22}H_2)$$

$$a'_{116} = (-Q_{122}\xi + Q_{222}\gamma_{22}\alpha_{22})\cosh(-\alpha_{22}H_2)$$

$$a'_{117} = 0, \quad a'_{118} = 0, \quad a'_{119} = 0, \quad a'_{1110} = 0$$

$$a'_{1111} = -(-Q_{124}\xi + Q_{224}\gamma_{14}\alpha_{14})\sinh(-\alpha_{14}H_2)$$

$$a'_{1112} = -(-Q_{124}\xi + Q_{224}\gamma_{14}\alpha_{14})\cosh(-\alpha_{14}H_2)$$

$$a'_{1113} = -(-Q_{124}\xi + Q_{224}\gamma_{24}\alpha_{24})\sinh(-\alpha_{24}H_2)$$

$$a'_{1114} = -(-Q_{124}\xi + Q_{224}\gamma_{24}\alpha_{24})\cosh(-\alpha_{24}H_2)$$

$$a'_{1115} = 0, \quad a'_{1116} = 0, \quad a'_{1117} = 0, \quad a'_{1118} = 0, \quad b'_{111} = 0, \quad b'_{112} = 0$$

$$\begin{aligned} b'_{113} = {} & (Q_{122}f_{12} + Q_{222}f_{22}/k_2 - \beta_{22})[f_2(\xi)\sinh(-|\xi|H_2/k_2) + f_3(\xi)\cosh(-|\xi|H_2/k_2)] \\ & - (Q_{124}f_{14} + Q_{224}f_{24}/k_4 - \beta_{24})[f_6(\xi)\sinh(-|\xi|H_2/k_4) + f_7(\xi)\cosh(-|\xi|H_2/k_4)] \end{aligned}$$

$$a'_{121} = 0, \quad a'_{122} = 0, \quad a'_{123} = Q_{662}(\alpha_{12} + \xi\gamma_{12})\cosh(-\alpha_{12}H_2)$$

$$a'_{124} = Q_{662}(\alpha_{12} + \xi\gamma_{12})\sinh(-\alpha_{12}H_2), \quad a'_{125} = Q_{662}(\alpha_{22} + \xi\gamma_{22})\cosh(-\alpha_{22}H_2)$$

$$a'_{126} = Q_{662}(\alpha_{22} + \xi\gamma_{22})\sinh(-\alpha_{22}H_2), \quad a'_{127} = 0, \quad a'_{128} = 0, \quad a'_{129} = 0, \quad a'_{1210} = 0$$

$$a'_{1211} = -Q_{664}(\alpha_{14} + \xi\gamma_{14})\cosh(-\alpha_{14}H_2), \quad a'_{1212} = -Q_{664}(\alpha_{14} + \xi\gamma_{14})\sinh(-\alpha_{14}H_2)$$

$$a'_{1213} = -Q_{664}(\alpha_{24} + \xi\gamma_{24})\cosh(-\alpha_{24}H_2), \quad a'_{1214} = -Q_{664}(\alpha_{24} + \xi\gamma_{24})\sinh(-\alpha_{24}H_2)$$

$$a'_{1215} = 0, \quad a'_{1216} = 0, \quad a'_{1217} = 0, \quad a'_{1218} = 0, \quad b'_{121} = 0, \quad b'_{122} = 0$$

$$\begin{aligned} b'_{123} = {} & -Q_{662}(f_{12}/k_2 - f_{22})[f_2(\xi)\cosh(-|\xi|H_2/k_2) + f_3(\xi)\sinh(-|\xi|H_2/k_2)]|\xi|/\xi \\ & + Q_{664}(f_{14}/k_4 - f_{24})[f_6(\xi)\cosh(-|\xi|H_2/k_4) + f_7(\xi)\sinh(-|\xi|H_2/k_4)]|\xi|/\xi \end{aligned}$$

$$a'_{131} = 0, \quad a'_{132} = 0, \quad a'_{133} = \sinh(-\alpha_{12}H_2), \quad a'_{134} = \cosh(-\alpha_{12}H_2)$$

$$a'_{135} = \sinh(-\alpha_{22}H_2), \quad a'_{136} = \cosh(-\alpha_{22}H_2), \quad a'_{137} = 0, \quad a'_{138} = 0, \quad a'_{139} = 0$$

$$a'_{1310} = 0, \quad a'_{1311} = \sinh(-\alpha_{14}H_2), \quad a'_{1312} = \cosh(-\alpha_{14}H_2), \quad a'_{1313} = \sinh(-\alpha_{24}H_2)$$

$$a'_{1314} = \cosh(-\alpha_{24}H_2), \quad a'_{1315} = 0, \quad a'_{1316} = 0, \quad a'_{1317} = 0, \quad a'_{1318} = 0, \quad b'_{131} = 0, \quad b'_{132} = 0$$

$$\begin{aligned} b'_{133} = {} & -f_{12}[f_2(\xi)\sinh(-|\xi|H_2/k_2) + f_3(\xi)\cosh(-|\xi|H_2/k_2)]/\xi \\ & + f_{14}[f_6(\xi)\sinh(-|\xi|H_2/k_4) + f_7(\xi)\cosh(-|\xi|H_2/k_4)]/\xi \end{aligned}$$

$$a'_{141} = 0, \quad a'_{142} = 0, \quad a'_{143} = \gamma_{12}\cosh(-\alpha_{12}H_2), \quad a'_{144} = \gamma_{12}\sinh(-\alpha_{12}H_2)$$

$$a'_{145} = \gamma_{22}\cosh(-\alpha_{22}H_2), \quad a'_{146} = \gamma_{22}\sinh(-\alpha_{22}H_2), \quad a'_{147} = 0, \quad a'_{148} = 0$$

$$a'_{149} = 0, \quad a'_{1410} = 0, \quad a'_{1411} = -\gamma_{14}\cosh(-\alpha_{14}H_2), \quad a'_{1412} = -\gamma_{14}\sinh(-\alpha_{14}H_2)$$

$$a'_{1413} = -\gamma_{24}\cosh(-\alpha_{24}H_2), \quad a'_{1414} = -\gamma_{24}\sinh(-\alpha_{24}H_2), \quad a'_{1415} = 0$$

$$a'_{1416} = 0, \quad a'_{1417} = 0, \quad a'_{1418} = 0, \quad b'_{141} = 0, \quad b'_{142} = 0$$

$$b'_{143} = f_{22}[f_3(\xi)\sinh(-|\xi|H_2/k_2) + f_2(\xi)\cosh(-|\xi|H_2/k_2)]/|\xi| \\ - f_{24}[f_7(\xi)\sinh(-|\xi|H_2/k_4) + f_6(\xi)\cosh(-|\xi|H_2/k_4)]/|\xi|$$

$$a'_{151} = 0, \quad a'_{152} = 0, \quad a'_{153} = 0, \quad a'_{154} = 0, \quad a'_{155} = 0, \quad a'_{156} = 0, \quad a'_{157} = 0$$

$$a'_{158} = 0, \quad a'_{159} = 0, \quad a'_{1510} = 0, \quad a'_{1511} = (-Q_{124}\xi + Q_{224}\gamma_{14}\alpha_{14})\sinh(-\alpha_{14}H_4)$$

$$a'_{1512} = (-Q_{124}\xi + Q_{224}\gamma_{14}\alpha_{14})\cosh(-\alpha_{14}H_4)$$

$$a'_{1513} = (-Q_{124}\xi + Q_{224}\gamma_{24}\alpha_{24})\sinh(-\alpha_{24}H_4)$$

$$a'_{1514} = (-Q_{124}\xi + Q_{224}\gamma_{24}\alpha_{24})\cosh(-\alpha_{24}H_4)$$

$$a'_{1515} = 0, \quad a'_{1516} = 0, \quad a'_{1517} = -(-Q_{126}\xi + Q_{226}\gamma_{16}\alpha_{16})\exp(-\alpha_{16}H_4)$$

$$a'_{1518} = -(-Q_{126}\xi + Q_{226}\gamma_{26}\alpha_{26})\exp(-\alpha_{26}H_4), \quad b'_{151} = 0, \quad b'_{152} = 0$$

$$b'_{153} = (Q_{124}f_{14} + Q_{224}f_{24}/k_4 - \beta_{24})[f_6(\xi)\sinh(-|\xi|H_4/k_4) + f_7(\xi)\cosh(-|\xi|H_4/k_4)] \\ - (Q_{126}f_{16} + Q_{226}f_{26}/k_6 - \beta_{26})f_9(\xi)\exp(-|\xi|H_4/k_6)$$

$$a'_{161} = 0, \quad a'_{162} = 0, \quad a'_{163} = 0, \quad a'_{164} = 0, \quad a'_{165} = 0, \quad a'_{166} = 0, \quad a'_{167} = 0$$

$$a'_{168} = 0, \quad a'_{169} = 0, \quad a'_{1610} = 0, \quad a'_{1611} = Q_{664}(\alpha_{14} + \xi\gamma_{14})\cosh(-\alpha_{14}H_4)$$

$$a'_{1612} = Q_{664}(\alpha_{14} + \xi\gamma_{14})\sinh(-\alpha_{14}H_4), \quad a'_{1613} = Q_{664}(\alpha_{24} + \xi\gamma_{24})\cosh(-\alpha_{24}H_4)$$

$$a'_{1614} = Q_{664}(\alpha_{24} + \xi\gamma_{24})\sinh(-\alpha_{24}H_4), \quad a'_{1615} = 0, \quad a'_{1616} = 0, \quad a'_{1617} = 0$$

$$a'_{1618} = 0, \quad b'_{161} = 0, \quad b'_{162} = 0$$

$$b'_{163} = -Q_{664}(f_{14}/k_4 - f_{24})[f_6(\xi)\cosh(-|\xi|H_4/k_4) + f_7(\xi)\sinh(-|\xi|H_4/k_4)]|\xi|/\xi \\ + Q_{666}(f_{16}/k_6 - f_{26})f_9(\xi)\exp(-|\xi|H_4/k_6)|\xi|/\xi$$

$$a'_{171} = 0, \quad a'_{172} = 0, \quad a'_{173} = 0, \quad a'_{174} = 0, \quad a'_{175} = 0, \quad a'_{176} = 0, \quad a'_{177} = 0$$

$$a'_{178} = 0, \quad a'_{179} = 0, \quad a'_{1710} = 0, \quad a'_{1711} = \sinh(-\alpha_{14}H_4), \quad a'_{1712} = \cosh(-\alpha_{14}H_4)$$

$$a'_{1713} = \sinh(-\alpha_{24}H_4), \quad a'_{1714} = \cosh(-\alpha_{24}H_4), \quad a'_{1715} = 0, \quad a'_{1716} = 0$$

$$a'_{1717} = 0, \quad a'_{1718} = 0, \quad b'_{171} = -\exp(-\alpha_{16}H_4), \quad b'_{172} = -\exp(-\alpha_{26}H_4)$$

$$b'_{173} = -f_{14}[f_6(\xi)\sinh(-|\xi|H_4/k_4) + f_7(\xi)\cosh(-|\xi|H_4/k_4)]/\xi + f_{16}f_9(\xi)\exp(-|\xi|H_4/k_6)/\xi$$

$$a'_{181} = 0, \quad a'_{182} = 0, \quad a'_{183} = 0, \quad a'_{184} = 0, \quad a'_{185} = 0, \quad a'_{186} = 0, \quad a'_{187} = 0$$

$$a'_{188} = 0, \quad a'_{189} = 0, \quad a'_{1810} = 0, \quad a'_{1811} = \gamma_{14}\cosh(-\alpha_{14}H_4), \quad a'_{1812} = \gamma_{14}\sinh(-\alpha_{14}H_4)$$

$$a'_{1813} = \gamma_{24}\cosh(-\alpha_{24}H_4), \quad a'_{1814} = \gamma_{24}\sinh(-\alpha_{24}H_4), \quad a'_{1815} = 0, \quad a'_{1816} = 0$$

$$a'_{1817} = 0, \quad a'_{1818} = 0, \quad b'_{181} = -\gamma_{16}\exp(-\alpha_{16}H_4), \quad b'_{182} = -\gamma_{26}\exp(-\alpha_{26}H_4),$$

$$b'_{183} = f_{24}[f_7(\xi)\sinh(-|\xi|H_4/k_4) + f_6(\xi)\cosh(-|\xi|H_4/k_4)]/|\xi| - f_{26}f_9(\xi)\exp(-|\xi|H_4/k_6)/|\xi| \tag{B.3}$$

Appendix C

$$\begin{aligned} Q_1(\xi) &= (l_4m_2 - l_5m_1)/\Delta'', \quad Q_2(\xi) = (l_1m_2 - l_2m_1)/\Delta'' \\ Q_3(\xi) &= \{[(l_3l_5 + l_6l_2)m_1 - (l_3l_4 + l_6l_1)m_2]/\Delta'' + m_3\} \\ Q_4(\xi) &= (l_5m_4 - l_4m_5)/\Delta'', \quad Q_5(\xi) = (l_2m_4 - l_1m_5)/\Delta'' \\ Q_6(\xi) &= \{[(l_3l_4 + l_6l_1)m_5 - (l_3l_5 + l_6l_2)m_4]/\Delta'' + m_6\} \end{aligned} \tag{C.1}$$

with

$$\begin{aligned} m_1 &= g_{21}(-Q_{121}\xi + Q_{221}\gamma_{21}\alpha_{21}), \\ m_2 &= (-Q_{121}\xi + Q_{221}\gamma_{11}\alpha_{11}) + g_{22}(-Q_{121}\xi + Q_{221}\gamma_{21}\alpha_{21}), \\ m_3 &= -g_{23}(-Q_{121}\xi + Q_{221}\gamma_{21}\alpha_{21}) + f_1(\xi)(Q_{121}f_{11} + Q_{221}f_{21}/k_1 - \beta_{21}), \\ m_4 &= Q_{661}[(\alpha_{11} + \xi\gamma_{11}) + g_{11}(\alpha_{21} + \xi\gamma_{21})], \quad m_5 = Q_{661}g_{12}(\alpha_{21} + \xi\gamma_{21}), \\ m_6 &= Q_{661}[g_{13}(\alpha_{21} + \xi\gamma_{21}) + (f_{11}/k_1 - f_{21})|\xi|/\xi \end{aligned} \tag{C.2}$$

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